

A BUNDLING PROBLEM REVISITED

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ABSTRACT. It was conjectured by M. Glasser and S. Davison and later proved by A. Eremenko that the certain animals should gather close to each other in order to decrease the total heat loss. In this paper we show that it is not always true for the individual heat loss. This gives a negative answer to a question posed by A. Eremenko.

1. MOTIVATION, MODEL AND RESULTS

In [2], [1] Glasser and Davison consider the following problem: let B_1 and B_2 are two disjoint balls in \mathbb{R}^3 with equal radii. Let $d \geq 0$ be a distance between the balls. Consider a harmonic function in the complement of the balls, i.e., $\mathbb{R}^3 \setminus B_1 \cup B_2$, such that $u|_{\partial B_j} = 1$ and $\lim_{|x| \rightarrow \infty} u(x) = 0$. Let

$$Q(d) = \int_{\partial B_1} \frac{\partial u}{\partial n} d\sigma + \int_{\partial B_2} \frac{\partial u}{\partial n} d\sigma$$

be the heat flux where n is the outward unit normal vector to a sphere. Is it true that the quantity $Q(d)$ is increasing as the function of distance between the balls?

The problem arises from the question why certain warm blooded animals like armadillos can keep each other warm by huddling together. In this simple model the balls B_1 and B_2 represent uniform spherical animals in \mathbb{R}^3 with body temperature 1 and medium temperature 0. The harmonic function $u(x)$ represents time independent temperature in $\mathbb{R}^3 \setminus B_1 \cup B_2$, and the quantity $Q(d)$ represents the total heat loss of both animals: the total amount of heat given off by the animals as a function of distance between the balls. Presumably moving animals closer together decreases the heat loss $Q(d)$, and it was confirmed numerically in [2] that the quantity $Q(d)$ is increasing. However no mathematical proof was given until A. Eremenko [3] gave a rigorous proof in more general setting. In Eremenko's argument it was noticed that

$$(1) \quad \frac{4\pi}{Q} = \inf \{I(\mu) : \text{supp } \mu \subset \partial B_1 \cup \partial B_2, \mu(\partial B_1 \cup \partial B_2) = 1, \mu \geq 0\}$$

where

$$I(\mu) = \int \int \frac{d\mu(x)d\mu(y)}{|x-y|}.$$

The monotonicity of Q follows from the fact that if $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is continuous, one-to-one, and $|\varphi(x) - \varphi(y)| \geq |x - y|$ then $I(\varphi_*\mu) \leq I(\mu)$ where $\varphi_*\mu(A) = \mu(\varphi^{-1}(A))$ for any Borel measurable $A \subset \mathbb{R}^3$.

Notice that each individual animal B_j feels only his own heat loss $Q_j(d)$

$$Q_j(d) = \int_{\partial B_j} \frac{\partial u}{\partial n} d\sigma,$$

but not the total $Q = Q_1 + Q_2$. Therefore the behavior of the animals we have discussed could be driven by individual feelings but not the abstract "common goal". In case of equal balls the individual heat loss Q_j is monotonically increasing because $Q_1 = Q_2 = \frac{Q}{2}$. It is natural to think that the individual heat loss Q_j is monotonically increasing for the balls of different radii. In [3, 4] the following question was asked:

Question. *Are the quantities $Q_j(d)$, $j = 1, 2$ monotonically increasing if the balls have different radii?*

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In Section 2 we will show that if r_j denotes the radius of the ball B_j for $j = 1, 2$, then the of heat loss Q_1 is not monotonically increasing provided that $\frac{r_1}{r_2} > \ell$ where $\ell \approx 1.95$ is the positive solution of the equation

$$(2) \quad 2(1+x^3) \left(\gamma + \psi \left(\frac{1}{1+x} \right) \right) + x^2 + x + 2(x^2 - x) \psi' \left(\frac{1}{1+x} \right) = 0.$$

In the above notation γ is the Euler's constant, and ψ is the digamma function.

In other words, if armadillo A is at least twice as big as armadillo B, then A should keep some nonzero distance from B in order to minimize the heat loss while B should try to be as close as possible to A.

In Section 2 we obtain the following asymptotic expression for $Q_1(d)$ which implies the conclusion (2). Let $r_1 > r_2 > 0$, $d \geq 0$, and let the temperature of the balls be constant and equal to $T_0 > 0$. Then

$$\begin{aligned} \frac{Q_1(d)}{4\pi T_0 r_1} = & -\frac{r_2 \left(\gamma + \psi \left(\frac{r_2}{r_1+r_2} \right) \right)}{r_1 + r_2} - \\ & \frac{d}{6r_1(r_1+r_2)^3} \left[2(r_1^3 + r_2^3) \left(\gamma + \psi \left(\frac{r_2}{r_1+r_2} \right) \right) + r_1^2 r_2 + r_2^2 r_1 + 2(r_1^2 r_2 - r_2^2 r_1) \psi' \left(\frac{r_2}{r_1+r_2} \right) \right] + o(d). \end{aligned}$$

as $d \rightarrow 0$.

2. TWO BALLS OF UNEQUAL RADII

We consider a bispherical coordinate system

$$(x, y, z) = \left(\frac{a \sin \eta \cos \phi}{\cosh \mu - \cos \eta}, \frac{a \sin \eta \sin \phi}{\cosh \mu - \cos \eta}, \frac{a \sinh \mu}{\cosh \mu - \cos \eta} \right),$$

where $-\infty \leq \mu \leq \infty$, $0 \leq \eta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $a > 0$ so that the foci F_1 and F_2 coincide with the centers of B_1 and B_2 . A general solution of the Laplace equation in bispherical coordinate system (under the assumptions that the solution does not depend on ϕ which is true in our case) is given by the expression (see page 1298, [6])

$$T(\mu, \eta) = \sqrt{\cosh \mu - \cos \eta} \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\mu} + B_n e^{-(n+\frac{1}{2})\mu} \right] P_n(\cos \eta),$$

where P_n are Legendre polynomials. Let $r_1 > 0$ and $r_2 > 0$ be the radii of B_1 and B_2 correspondingly. The corresponding values of the μ coordinate $\mu = \mu_1 > 0$ on ∂B_1 and $\mu = \mu_2 < 0$ on ∂B_2 will be determined by $r_j = \frac{a}{|\sinh \mu_j|}$ for $j = 1, 2$. Notice also that the distance d between the balls can be obtained as follows

$$d + r_1 + r_2 = a(\coth \mu_1 - \coth \mu_2) = r_1 \cosh \mu_1 + r_2 \cosh \mu_2.$$

Assume that the temperature on ∂B_j is a constant T_j , $j = 1, 2$. Then by using the generating function for the Legendre polynomials

$$(3) \quad \frac{1}{\sqrt{\cosh \mu - \cos \eta}} = \sqrt{2} \sum_{n=0}^{\infty} e^{-(n+\frac{1}{2})|\mu|} P_n(\cos \eta),$$

the boundary condition $T(\mu_j, \eta) = T_j$ implies

$$(4) \quad A_n e^{(n+\frac{1}{2})\mu_j} + B_n e^{-(n+\frac{1}{2})\mu_j} = T_j \sqrt{2} e^{-(n+\frac{1}{2})|\mu_j|}, \quad j = 1, 2 \quad n \geq 0.$$

Let us compute the surface element $d\sigma$ in the bispherical coordinates:

$$|(x, y, z)_\eta \times (x, y, z)_\phi| = \frac{a^2 \sin \eta}{(\cosh \mu - \cos \eta)^2}.$$

The heat loss Q_1 can be computed as follows

$$Q_1 = \iint_{\partial B_1} \frac{\partial T}{\partial n} \Big|_{\mu=\mu_1} \frac{a^2 \sin \eta d\eta d\phi}{(\cosh \mu_1 - \cos \eta)^2}.$$

Notice that on ∂B_1 we have

$$\begin{aligned}
 \frac{\partial T}{\partial n} &= \left(\frac{\cosh \mu_1 - \cos \eta}{a} \right) \frac{\partial T}{\partial \mu} = \\
 &= \frac{\sinh \mu_1}{2a} \sqrt{\cosh \mu_1 - \cos \eta} \sum_{n=0}^{\infty} \left[A_n e^{(n+\frac{1}{2})\mu_1} + B_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n(\cos \eta) + \\
 &= \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{a} \sum_{n=0}^{\infty} \left[\left(n + \frac{1}{2} \right) A_n e^{(n+\frac{1}{2})\mu_1} - \left(n + \frac{1}{2} \right) B_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n(\cos \eta) = \\
 (5) \quad &= \frac{\sinh \mu_1}{2a} T(\mu_1, \eta) + \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{a} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) \left[A_n e^{(n+\frac{1}{2})\mu_1} - B_n e^{-(n+\frac{1}{2})\mu_1} \right] P_n(\cos \eta).
 \end{aligned}$$

The boundary conditions (4) imply that

$$\begin{aligned}
 A_n e^{(n+\frac{1}{2})\mu_1} - B_n e^{-(n+\frac{1}{2})\mu_1} &= \sqrt{2} \frac{e^{-(n+\frac{1}{2})\mu_1} T_1 + e^{-(n+\frac{1}{2})(3\mu_1-2\mu_2)} T_1 - 2T_2 e^{-(n+\frac{1}{2})(\mu_1-2\mu_2)}}{1 - e^{-(n+\frac{1}{2})(2\mu_1-2\mu_2)}} = \sum_{k=0}^{\infty} \\
 &= \left(T_1 \sqrt{2} e^{-(n+\frac{1}{2})(\mu_1+k(2\mu_1-2\mu_2))} + T_1 \sqrt{2} e^{-(n+\frac{1}{2})(3\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} - T_2 2\sqrt{2} e^{-(n+\frac{1}{2})(\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} \right) = \\
 &= T_1 \sqrt{2} e^{-(n+\frac{1}{2})\mu_1} + 2\sqrt{2} \sum_{k=0}^{\infty} \left(T_1 e^{-(n+\frac{1}{2})(3\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} - T_2 e^{-(n+\frac{1}{2})(\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} \right).
 \end{aligned}$$

Thus (5) takes the following form

$$\begin{aligned}
 \frac{\partial T}{\partial n} &= \frac{\sinh \mu_1}{2a} T(\mu_1, \eta) + \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{a} T_1 \sqrt{2} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) e^{-(n+\frac{1}{2})\mu_1} P_n(\cos \eta) + \\
 &= \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{a} \cdot 2\sqrt{2} T_1 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(n + \frac{1}{2} \right) e^{-(n+\frac{1}{2})(3\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} P_n(\cos \eta) \\
 &\quad - \frac{(\cosh \mu_1 - \cos \eta)^{3/2}}{a} \cdot 2\sqrt{2} T_2 \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(n + \frac{1}{2} \right) e^{-(n+\frac{1}{2})(\mu_1-2\mu_2+k(2\mu_1-2\mu_2))} P_n(\cos \eta).
 \end{aligned}$$

If we differentiate (3) with respect to μ we obtain the following identity

$$(6) \quad \frac{1}{2} \sinh \mu (\cosh \mu - \cos \eta)^{-3/2} = \sqrt{2} \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right) e^{-(n+\frac{1}{2})\mu} P_n(\cos \eta) \quad \text{for } \mu > 0.$$

Using (6) we further simplify the expression for $\frac{\partial T}{\partial n}$:

$$\begin{aligned}
 \frac{\partial T}{\partial n} &= \frac{\sinh \mu_1}{2a} T(\mu_1, \eta) + \frac{\sinh \mu_1}{2a} T_1 + \\
 &= \sum_{k=0}^{\infty} \frac{T_1}{a} \sinh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \left(\frac{\cosh \mu_1 - \cos \eta}{\cosh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - \cos \eta} \right)^{3/2} \\
 &\quad - \sum_{k=0}^{\infty} \frac{T_2}{a} \sinh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \left(\frac{\cosh \mu_1 - \cos \eta}{\cosh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - \cos \eta} \right)^{3/2}
 \end{aligned}$$

We notice that $T(\mu_1, \eta) = T_1$. Then $\frac{\sinh \mu_1}{2a} T(\mu_1, \eta) + \frac{\sinh \mu_1}{2a} T_1 = \frac{\sinh \mu_1}{a} T_1$ and finally we have

$$Q_1 = \int_0^\pi \left[2\pi a T_1 \sinh \mu_1 \frac{\sin \eta}{(\cosh \mu_1 - \cos \eta)^2} + \sum_{k=0}^{\infty} \left(2\pi a T_1 \sinh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \frac{(\cosh \mu_1 - \cos \eta)^{-1/2} \sin \eta}{(\cosh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - \cos \eta)^{3/2}} - \right. \right. \\ \left. \left. 2\pi a T_2 \sinh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \frac{(\cosh \mu_1 - \cos \eta)^{-1/2} \sin \eta}{(\cosh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - \cos \eta)^{3/2}} \right) \right] d\eta.$$

By substituting $\cos \eta = x$ we obtain

$$Q_1 = \int_{-1}^1 \left[2\pi a T_1 \sinh \mu_1 \frac{1}{(\cosh \mu_1 - x)^2} + \sum_{k=0}^{\infty} \left(2\pi a T_1 \sinh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \frac{(\cosh \mu_1 - x)^{-1/2}}{(\cosh(3\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - x)^{3/2}} - \right. \right. \\ \left. \left. 2\pi a T_2 \sinh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) \cdot \frac{(\cosh \mu_1 - x)^{-1/2}}{(\cosh(\mu_1 - 2\mu_2 + k(2\mu_1 - 2\mu_2)) - x)^{3/2}} \right) \right] dx = \\ I_1 + \sum_{k=0}^{\infty} (I_{2,k} + I_{3,k}).$$

Let us calculate each term separately. We remind that $\frac{\sinh \mu_1}{2a} T_1 = \frac{T_1}{2r_1}$. Therefore we have

$$I_1 = \frac{4\pi a T_1 \sinh \mu_1}{\sinh^2 \mu_1} = 4\pi T_1 r_1.$$

We notice the following subtle identity

$$(7) \quad \sinh(B) \int_{-1}^1 \frac{(\cosh(A) - x)^{-1/2} dx}{(\cosh(B) - x)^{3/2}} = \frac{2}{\sinh\left(\frac{A+B}{2}\right)}$$

for all real numbers A and B whenever the both sides of (7) make sense. Then

$$I_{2,k} = \frac{4\pi a T_1}{\sinh(2\mu_1 - \mu_2 + k(\mu_1 - \mu_2))}, \\ I_{3,k} = -\frac{4\pi a T_2}{\sinh(\mu_1 - \mu_2 + k(\mu_1 - \mu_2))}.$$

Taking into account that $r_1 = \frac{a}{\sinh \mu_1}$ we obtain

$$Q_1 = 4\pi T_1 r_1 + 4\pi a \sum_{k=0}^{\infty} \frac{T_1}{\sinh(2\mu_1 - \mu_2 + k(\mu_1 - \mu_2))} - \frac{T_2}{\sinh(\mu_1 - \mu_2 + k(\mu_1 - \mu_2))} = \\ 4\pi T_1 r_1 + 4\pi r_1 \sinh(\mu_1) \sum_{k=1}^{\infty} \frac{T_1}{\sinh(\mu_1 + k(\mu_1 - \mu_2))} - \frac{T_2}{\sinh(k(\mu_1 - \mu_2))}.$$

Further we consider the case when $T_1 = T_2 = T_0 > 0$. In order to investigate the monotonicity of Q_1 with respect to d , it is enough to investigate the monotonicity of the following function

$$f(d) := \sinh(\mu_1) \sum_{k=1}^{\infty} \frac{1}{\sinh(\mu_1 + k(\mu_1 - \mu_2))} - \frac{1}{\sinh(k(\mu_1 - \mu_2))}.$$

We notice that

$$\cosh \mu_1 = \frac{(d+r_1+r_2)^2 + r_1^2 - r_2^2}{2(d+r_1+r_2)r_1} \quad \text{and} \quad \cosh \mu_2 = \frac{(d+r_1+r_2)^2 + r_2^2 - r_1^2}{2(d+r_1+r_2)r_2}.$$

Let $x = e^{\mu_1} > 1$ and $y = e^{-\mu_2} > 1$. Then $f(d)$ takes the form

$$(8) \quad f(d) = \left(x - \frac{1}{x}\right) \sum_{k=1}^{\infty} \left(\frac{x(xy)^k}{x^2(xy)^{2k} - 1} - \frac{(xy)^k}{(xy)^{2k} - 1} \right)$$

where

$$(9) \quad x = 1 + \frac{d(d+2r_2) + \sqrt{(d+2r_1+2r_2)(d+2r_2)(d+2r_1)d}}{2(d+r_1+r_2)r_1};$$

$$(10) \quad y = 1 + \frac{d(d+2r_1) + \sqrt{(d+2r_1+2r_2)(d+2r_2)(d+2r_1)d}}{2(d+r_1+r_2)r_2}.$$

By using the identity $\frac{1}{t-1} = \frac{1}{t} \sum_{k=0}^{\infty} t^{-k}$ two times for the terms inside the summation (8), and by Fubini's theorem the expression for $f(d)$ can be simplified as follows

$$f(d) = \left(x - \frac{1}{x}\right) \sum_{j=0}^{\infty} \frac{x^{-2j-1} - 1}{(xy)^{2j+1} - 1}.$$

If $r_1 = r_2$ then, as we already mentioned in (1), it is known that f is monotonically increasing (see also a proof in [5] without resorting to (1)). Therefore it is enough to study the sign of $\lim_{d \rightarrow 0+} f'(d)$ for different radii r_1 and r_2 . Further we assume that $r_1 > r_2$. Let $z = xy$. Set

$$f(d) = \left(x - \frac{1}{x}\right) \sum_{j=0}^{\infty} \frac{x^{-2j-1} - 1}{z^{2j+1} - 1} = \left(x - \frac{1}{x}\right) \sum_{k=0}^{\infty} g(k),$$

where $g(s) = \frac{x^{-2s-1} - 1}{z^{2s+1} - 1}$. Since $x, z > 1$ it is easy to see that $g(s) \in C^\infty([0, \infty))$ and all its derivatives tend to zero as $k \rightarrow \infty$. Therefore By Euler–Maclaurin formula we have

$$(11) \quad \left(x - \frac{1}{x}\right) \sum_{k=0}^{\infty} g(k) = \left(x - \frac{1}{x}\right) \left(\int_0^\infty g(s) ds + \frac{g(0)}{2} - \frac{g'(0)}{12} \right) - \left(x - \frac{1}{x}\right) \int_0^\infty \frac{B_2(\{1-s\})}{2} g^{(2)}(s) ds$$

where $B_2(x) = x^2 - x + \frac{1}{2}$ is the Bernoulli polynomial, and $\{x\}$ represents the fractional part of x .

We will compute the asymptotic behavior of each term in (11) separately as $t \rightarrow 0$. First notice that (9) implies

$$(12) \quad x - \frac{1}{x} = 2\sqrt{\frac{2r_2}{(r_1+r_2)r_1}} d^{1/2} + \frac{1}{\sqrt{2}} \frac{r_1^2 - r_1 r_2 + r_2^2}{(r_1+r_2)^{3/2} \sqrt{r_2} r_1^{3/2}} d^{3/2} + O(d^2).$$

We have

$$\int_0^\infty g(t) dt = \int_0^\infty \frac{x^{-1} e^{-2t \ln x} - 1}{z e^{2t \ln z} - 1} dt = \frac{1}{\ln z} \int_0^\infty \frac{x^{-1} e^{-s \frac{\ln x}{\ln z}} - 1}{z e^s - 1} ds.$$

We notice that

$$\int_0^\infty \frac{1}{z e^s - 1} ds = \ln \left(\frac{z}{z-1} \right).$$

Therefore

$$\frac{1}{\ln z} \int_0^\infty \frac{x^{-1} e^{-s \frac{\ln x}{\ln z}} - 1}{z e^s - 1} ds = -\frac{\ln \left(\frac{z}{z-1} \right)}{2 \ln z} + \frac{1}{2x \ln z} \int_0^\infty \frac{e^{-s \frac{\ln x}{\ln z}}}{z e^s - 1} ds.$$

Set $\alpha = \frac{\ln x}{\ln z}$. Then

$$\begin{aligned} \int_0^\infty \frac{e^{-s\alpha}}{ze^s - 1} ds &= \int_1^\infty y^{-\alpha} \left[-\frac{1}{y} + \frac{z}{zy - 1} \right] dy = -\frac{1}{\alpha} + z \int_1^\infty \frac{y^{-\alpha}}{zy - 1} dy = \\ &= -\frac{1}{\alpha} + x \int_0^{1/z} \frac{s^{\alpha-1}}{1-s} ds = \frac{1}{z} \sum_{k=0}^\infty \frac{(z^{-1})^k}{\alpha + 1 + k} = \frac{z^{-1}}{1+\alpha} {}_2F_1(1, 1+\alpha; 2+\alpha; z^{-1}), \end{aligned}$$

where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

denotes the hypergeometric function where $(a)_n = a(a+1) \cdots (a+n-1)$ if $n \geq 1$ and $(a)_0 = 1$. On the other hand it is known that (see [7])

$${}_2F_1(a, b; a+b; v) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\sum_{k=0}^\infty \frac{(a)_k (b)_k}{k!^2} (-\ln(1-v) + 2\psi(k+1) - \psi(a+k) - \psi(b+k))(1-v)^k \right)$$

for all $0 < 1-v < 1$, where ψ is the digamma function. Therefore we obtain

$$\begin{aligned} \frac{1}{2x \ln z} \int_0^\infty \frac{e^{-s \frac{\ln x}{\ln z}}}{ze^s - 1} ds &= \frac{1}{2xz \ln z} \frac{{}_2F_1(1, 1+\alpha; 2+\alpha; z^{-1})}{1+\alpha} = \\ &= \frac{1}{2xz \ln z} \sum_{k=0}^\infty \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right). \end{aligned}$$

Note that when $d \rightarrow 0+$ we have $\alpha = \frac{r_2}{r_1+r_2} + O(\sqrt{d})$, $1-z^{-1} = \sqrt{\frac{2(r_1+r_2)}{r_1 r_2}} d^{1/2} + O(d)$ and $\ln(1-z^{-1})(1-z^{-1})^3 = O(d^{3/2} \ln d)$. It is known that $\psi(t) = \ln(t) + O(1/t)$ for $t \rightarrow \infty$. Therefore if d is sufficiently small we have $\alpha < 1$ and thus for $k \geq 3$ we obtain

$$\left| \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right) \right| \leq 10kb(r_1, r_2) |\ln d| (c(r_1, r_2) d^{1/2})^k,$$

where $b(r_1, r_2) > 0$ and $c(r_1, r_2) > 0$ are some finite numbers depending on r_1 and r_2 . Therefore for sufficiently small d we have

$$\begin{aligned} \left| \sum_{k=3}^\infty \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right) \right| &\leq \\ 10b(r_1, r_2) |\ln d| \sum_{k=3}^\infty k(c(r_1, r_2) d^{1/2})^k &\leq 100b(r_1, r_2) |\ln d| c(r_1, r_2)^3 d^{3/2} = O(d^{3/2} \ln d). \end{aligned}$$

We obtain

$$\begin{aligned} \sum_{k=0}^\infty \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right) &= \\ \sum_{k=0}^2 \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right) &+ O(d^{3/2} \ln d). \end{aligned}$$

Notice that $(x - \frac{1}{x})\frac{1}{2xz\ln z} = \frac{r_2}{r_1+r_2} + O(\sqrt{d})$. Thus we obtain

$$\begin{aligned} & \left(x - \frac{1}{x}\right) \left[\int_0^\infty g(s)ds + \frac{g(0)}{2} - \frac{g'(0)}{12} \right] = \left(x - \frac{1}{x}\right) \left[\frac{g(0)}{2} - \frac{g'(0)}{12} - \frac{\ln\left(\frac{z}{z-1}\right)}{2\ln z} + \right. \\ & \left. \frac{1}{2xz\ln z} \left(O(d^{3/2}\ln d) + \sum_{k=0}^2 \left(\frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right) \right) \right] = \\ & \left(x - \frac{1}{x}\right) \left[\frac{x^{-1}-1}{2(z-1)} + \frac{z\ln x - \ln x + z\ln z - zx\ln z}{6x(z-1)^2} - \frac{\ln\left(\frac{z}{z-1}\right)}{2\ln z} + \right. \\ & \left. \frac{1}{2xz\ln z} \sum_{k=0}^2 \frac{(1+\alpha)_k}{k!} (-\ln(1-z^{-1}) + \psi(k+1) - \psi(1+k+\alpha))(1-z^{-1})^k \right] + O(d^{3/2}\ln d) \end{aligned}$$

We note that $\psi(1) = -\gamma$, then one can check that when $d \rightarrow 0$, after some routine computations, using (9), (10) and the identity $\psi(x+1) = \psi(x) + \frac{1}{x}$ several times, the above expression takes the following form

$$\begin{aligned} & - \frac{r_2 \left(\gamma + \psi\left(\frac{2r_2+r_1}{r_1+r_2}\right) \right)}{r_1+r_2} - \\ & \frac{d}{6r_1(r_1+r_2)^3} \left[(r_1^3 + r_2^3) \left(2\gamma + 2\psi\left(\frac{r_2}{r_1+r_2}\right) \right) + r_1^2 r_2 + r_2^2 r_1 + 2(r_1^2 r_2 - r_2^2 r_1) \psi'\left(\frac{r_2}{r_1+r_2}\right) \right] + o(d). \end{aligned}$$

We are left with showing that the term $(x - \frac{1}{x}) \int_0^\infty \frac{B_2(\{1-s\})}{2} g^{(2)}(s)ds$ in (11) behaves as $o(d)$ for sufficiently small d . Since $x - \frac{1}{x} = O(\sqrt{d})$ it is enough to show that $\int_0^\infty \frac{B_2(\{1-s\})}{2} g^{(2)}(s)ds = o(\sqrt{d})$. We have

$$(13) \quad \left| \int_0^\infty B_2(\{1-s\}) g^{(2)}(s)ds \right| = \frac{1}{3} \left| \int_0^\infty B_3(\{1-s\}) g^{(3)}(s)ds \right| \leq \int_0^\infty |g^{(3)}(s)|ds$$

where $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ is Bernoulli polynomial. Consider a function $f(s) = \frac{e^{sp}-1}{e^s-1}$ where $p = -\frac{\ln x}{\ln xy}$. Clearly $f((2s+1)\ln xy) = g(s)$. We need the following technical lemma:

Lemma 1. *Let $-1 \leq p \leq 0$. Then $f^{(3)}(s) \geq 0$ for all $s \geq 0$.*

Before we proceed to the proof of the lemma we will show how the desired estimate follows from the lemma. First notice that $p = -\frac{\ln x}{\ln xy} \in [-1, 0]$ because $x, y > 1$. Therefore $\text{sgn}(g^{(3)}) = \text{sgn}(f^{(3)}) > 0$ for all $s > 0$, and we have

$$\int_0^\infty |g^{(3)}(s)|ds = -g''(0).$$

After some straightforward computations one can show that $g''(0) = O(d)$ as $d \rightarrow 0$. We will omit the details of the unnecessary computations.

Finally, we obtain that

$$\begin{aligned} \frac{Q_1(d)}{4\pi T_0 r_1} &= 1 - \frac{r_2 \left(\gamma + \psi\left(\frac{2r_2+r_1}{r_1+r_2}\right) \right)}{r_1+r_2} - \\ & \frac{d}{6r_1(r_1+r_2)^3} \left[(r_1^3 + r_2^3) \left(2\gamma + 2\psi\left(\frac{r_2}{r_1+r_2}\right) \right) + r_1^2 r_2 + r_2^2 r_1 + 2(r_1^2 r_2 - r_2^2 r_1) \psi'\left(\frac{r_2}{r_1+r_2}\right) \right] + o(d) = \\ & - \frac{r_2 \left(\gamma + \psi\left(\frac{r_2}{r_1+r_2}\right) \right)}{r_1+r_2} - \\ & \frac{d}{6r_1(r_1+r_2)^3} \left[(r_1^3 + r_2^3) \left(2\gamma + 2\psi\left(\frac{r_2}{r_1+r_2}\right) \right) + r_1^2 r_2 + r_2^2 r_1 + 2(r_1^2 r_2 - r_2^2 r_1) \psi'\left(\frac{r_2}{r_1+r_2}\right) \right] + o(d) \end{aligned}$$

where $T_0 = T_1 = T_2$ is the temperature of the balls.

It remains to prove the technical lemma.

Proof. Notice that

$$(14) \quad f^{(3)}(s) = \frac{e^{s(p+3)}(p-1)^3 + e^{s(p+2)}(-3p^3 + 6p^2 - 4) + e^{s(p+1)}(3p^3 - 3p^2 - 3p - 1) - p^3 e^{sp} + e^{3s} + 4e^{2s} + e^s}{(e^s - 1)^4}$$

It is enough to show that the coefficient a_k of $\frac{s^k}{k!}$ for $k \geq 0$ of numerator in (14) is nonnegative. Indeed

$$a_k = (p+3)^k(p-1)^3 + (p+2)^k(-3p^3 + 6p^2 - 4) + (p+1)^k(3p^3 - 3p^2 - 3p - 1) - p^{k+3} + 3^k + 4 \cdot 2^k + 1 = J_{1,k}(p) + J_{2,k}(p) + J_{3,k}(p),$$

where

$$\begin{aligned} J_{1,k} &= (p+1)^k(3p^3 - 3p^2 - 3p - 1) + 1; \\ J_{2,k} &= (p+2)^k(-3p^3 + 6p^2) - p^{k+3} \\ J_{3,k} &= (p+3)^k(p-1)^3 - 4(p+2)^k + 3^k + 4 \cdot 2^k; \end{aligned}$$

We notice that $a_0 = a_1 = a_2 = a_3 = 0$, $a_4 = 6p^2(p-1)^2 \geq 0$ and $a_5 = 4p(p-1)(6p^2(p+1) - 14p + 1) \geq 0$ because of the assumptions on p . It is enough to show that $J_{1,k}, J_{2,k}$ and $J_{3,k}$ are nonnegative for all $k \geq 6$. Indeed, since $p \in [-1, 0]$ we have

$$J_{1,k}(p) = (p+1)^k(3p^3 - 3p^2 - 3p - 1) + 1 \geq (p+1)(3p^3 - 3p^2 - 3p - 1) + 1 = p(3p^3 - 6p - 4) \geq 0$$

For $J_{2,k}$ if k is even then $-p^{k+3} \geq 0$ and there is nothing to prove. Assume that $k = 2m + 1$ where $m \geq 3$. Then

$$J_{2,k} = (p+2)^{2m+1}(-3p^3 + 6p^2) - p^{2m+4} \geq (p+2)(-3p^3 + 6p^2) - p^4 = 4p^2(3 - p^2) \geq 0$$

It remains to show that $J_{3,k}(p) \geq 0$. We have $J_{3,k}(0) = 0$, and

$$\begin{aligned} J'_{3,k}(p) &= k(p+3)^{k-1}(p-1)^3 + 3(p+3)^k(p-1)^2 - 4k(p+2)^{k-1} = \\ &= (k+3)(p+2)^{k-1} \left[\left(1 + \frac{1}{p+2}\right)^{k-1}(p-1)^2 \left(p + \frac{9-k}{3+k}\right) - \frac{4k}{k+3} \right]. \end{aligned}$$

If $k \geq 9$ there is nothing to prove. We assume $k = 6, 7$ and 8 . In this case the only interesting situation is when $\frac{k-9}{3+k} \leq p \leq 0$. Then

$$\begin{aligned} \left(1 + \frac{1}{p+2}\right)^{k-1}(p-1)^2 \left(p + \frac{9-k}{3+k}\right) - \frac{4k}{k+3} &\leq \left(\frac{3}{2}\right)^{k-1} (p-1)^2 \left(p + \frac{9-k}{3+k}\right) - \frac{4k}{k+3} = \\ \left(\frac{3}{2}\right)^{k-1} \left[p^3 - \frac{3(k-1)}{k+3} p^2 + \frac{3(k-5)}{k+3} p + \frac{1}{k+3} \left(9 - k - 4k \left(\frac{2}{3}\right)^{k-1} \right) \right] &\leq 0. \end{aligned}$$

The last inequality follows because the signs of the coefficients of the polynomial alternate, and $p \leq 0$. \square

3. CONCLUSIONS

It follows from the previous section that if $r_1 > \ell r_2$ (where $\ell \approx 1.95$ is a positive solution of (2)) then the heat loss of the big ball, i.e., $Q_1(d)$ is decreasing when $d \in [0, \varepsilon]$ where $\varepsilon > 0$ is sufficiently small. Notice that when $d \rightarrow \infty$ we have $x \approx \frac{d}{r_1}$, and $y \approx \frac{d}{r_2}$ therefore $\lim_{d \rightarrow \infty} f(d) = 0$ and thus

$$Q_1(\infty) = 4\pi T_0 r_1 > -4\pi T_0 r_1 \left(\frac{r_2(\gamma + \psi(\frac{r_2}{r_1+r_2}))}{r_1+r_2} \right) = Q_1(0).$$

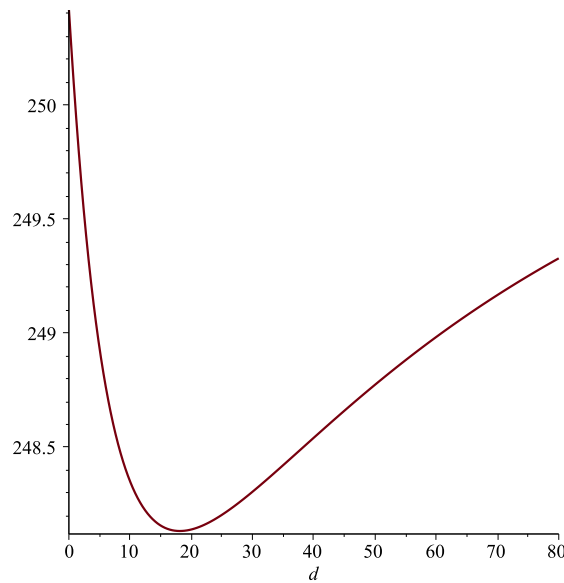


FIGURE 1. The heat loss $Q_1(d)$ of the big ball. $T_0 = 1, r_1 = 20, r_2 = 1, 0 \leq d \leq 80$

The last inequality is justified because $1 > -x(\gamma + \psi(x))$ for $x \in (0, 1)$ follows from the fact that $\psi(x+1) > -\gamma$ for $x \in (0, \infty)$. So there exists a minimal value of $Q_1(d)$ on $[0, \infty)$, i.e., a nonzero distance when the heat loss of the big ball is minimal.

The numerical computations show that, in fact $Q_1(d)$ is decreasing on the interval $[0, c)$ and then it is increasing on (c, ∞) where $c \approx r_1$. The heat loss of the small ball $Q_2(d)$ is always increasing for $d \geq 0$. Figure 1 represents the graph of $Q_1(d)$ where $T_0 = 1, r_1 = 20, r_2 = 1$ and $0 \leq d \leq 80$.

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